Journal of Engineering Mathematics **27**: 209–219, 1993.

Stokes drag on a disk sedimenting edgewise toward a plane wall

A.M.J. DAVIS
Mathematics Department, The University of Alabama, Tuscaloosa, AL 35487-0350, USA

Received 26 March 1992; accepted in revised form 6 October 1992

Abstract. For comparison with the results of current experiments, a concise analytical method is developed to describe the non-axisymmetric flow generated by the edgewise sedimentation of a disk toward a plane wall. Since all Fourier modes contribute to the flow, the use of Abel transforms in earlier work must be suitably extended in order to again obtain integral equations of the second kind. By expanding the wall effects in powers of D^{-1}, where D is the distance from the disk axis to the wall, the dimensionless drag coefficient is found to order D^{-5} without having to solve the flow problem beyond the second Fourier mode.

1. Introduction

An inevitable consequence of the slow rate of decay of creeping flow disturbances is that experimental results must invariably be modified by correction factors that take account of necessary, relatively distant boundaries in the laboratory. Thus, in measuring the drag on a disk moving edgewise between parallel walls, allowance must be made for the remaining container walls as must be similarly done when measuring the drag on a disk moving sideways towards a plane wall. This latter situation has potential engineering applications and is currently the subject of experimental work by Trahan and Hussey [1]. Here the effect of the wall is to increase from one to infinity the number of Fourier modes, with respect to the axis of the disk, that can contribute to the flow. Thus it is of interest to investigate whether the analytical methods, involving Abel transforms and integral equations, can be suitably extended from earlier work.

Attempts to establish a solvable, infinite set of dual integral equations, by using the velocity representation given by Ranger [2], as for the disk in isolation [3], or the shear flow past a hole in a plane [4, 5] were unsuccessful so a more basic procedure is adopted here. After expressing the velocity field as that due to a distribution of tangentially directed stokeslets, modified by the presence of the wall, over the disk, a pair of two-dimensional integral equations of the first kind for the unknown density functions is obtained. Then the structure of the double Fourier expansion of the singular terms in the kernels suggests the introduction of Abel transforms, as in [6], after which a procedure equivalent to inverting an Abel integral equation yields an infinite system of integral equations of the second kind. By this stage, the terms involving the reflected velocities, have become more complicated and no longer available in closed form. This major difficulty is conveniently overcome by seeking a solution in inverse powers of D, the distance between the disk axis and the wall. The separable structure of the kernels then identifies the Abel transforms of the density components to be even or odd polynomials of increasing order in D^{-1}, after which the solution is elementary. The dimensionless drag coefficient is evaluated to order D^{-5} in equation (25) without having to solve the integral equations beyond the second Fourier mode. This latter contributes at order D^{-3} whereas the first mode does so at order D^{-4}. The presentation below is designed to emphasize the structure of the solution involving all Fourier modes. An earlier decision to expand in powers of D^{-1}, together with some hindsight, would likely reduce the algebra by allowing some reordering of the steps.

2. Formulation of the disk problem

A thin rigid disk of unit radius translates steadily edgewise toward a rigid wall in incompressible viscous fluid that is at rest at infinity. Cylindrical polar coordinates (ρ, θ, z) are chosen so that the disk is instantaneously at $z = 0$ $(0 \leq \rho \leq 1,\ -\pi < \theta \leq \pi)$ and moving with velocity $U_0\hat{x}$ toward a rigid plane at $x = D$, where $(x, y) = \rho(\cos\theta, \sin\theta)$ and $\hat{x}, \hat{y}, \hat{z}$ denote unit vectors. The Reynolds number is assumed to be sufficiently small for the velocity field $\boldsymbol{v}$ to satisfy the creeping flow equations:

$$\mu \nabla^2 \boldsymbol{v} = \nabla p \,, \tag{1}$$

$$\nabla \cdot \boldsymbol{v} = 0 \,, \tag{2}$$

where μ is the coefficient of viscosity and p the dynamic pressure. The boundary conditions on the disk are

$$v_x = U_0,\ v_y = 0 = v_z \text{ on } z = 0\ (0 \leq \rho \leq 1)\,, \tag{3}$$

while no-slip at the plane requires

$$v_x = 0 = v_y = v_z \text{ on } x = D\,. \tag{4}$$

The only symmetries in the flow are with respect to the planes $y = 0$ and $z = 0$. Symmetry with respect to the plane of the disk implies that the flow can be represented as that due to distributions, over the disk, of tangentially directed stokeslets, modified to take account of the wall conditions (4). Symmetry with respect to the plane $y = 0$ implies that the distribution densities for stokeslets in the $\hat{x}$ and $\hat{y}$ directions are respectively even and odd functions of y, i.e. the polar angle. Hence, the velocity field $\boldsymbol{v}$, satisfying (1), (2) and (4) and such that $v_z = 0$ at $z = 0$, can be written in the form

$$\begin{aligned}
\boldsymbol{v} = {} & \frac{U_0}{\pi^2} \int_{-\pi}^{\pi} \int_0^1 f(\alpha, \phi) \left\{ \hat{x}\left[\frac{1}{r} + \frac{(x - \alpha\cos\phi)^2}{r^3}\right] + \frac{(x - \alpha\cos\phi)}{r^3} \right. \\
& \left. \times [(y - \alpha\sin\phi)\hat{y} + z\hat{z}] - V_x\hat{x} - V_y\hat{y} - V_z\hat{z} \right\} \alpha \, d\alpha \, d\phi \\
& + \frac{U_0}{\pi^2} \int_{-\pi}^{\pi} \int_0^1 g(\alpha, \phi) \left\{ \hat{y}\left[\frac{1}{r} + \frac{(y - \alpha\sin\phi)^2}{r^3}\right] + \frac{(y - \alpha\sin\phi)}{r^3} \right. \\
& \left. \times [(x - \alpha\cos\phi)\hat{x} + z\hat{z}] - U_x\hat{x} - U_y\hat{y} - U_z\hat{z} \right\} \alpha \, d\alpha \, d\phi \,,
\end{aligned} \tag{5}$$

where $r^2 = (x - \alpha\cos\phi)^2 + (y - \alpha\sin\phi)^2 + z^2$, f and g are respectively even and odd functions of α and the reflected velocity fields $\boldsymbol{V}$ and $\boldsymbol{U}$ are given, from [7], in terms of image stokeslets, stokes-doublets and source-doublets by

$$
\begin{aligned}
V_x &= \frac{2}{R} + \frac{4}{R^3}(D-x)(D-\alpha\cos\phi) \\
&\quad - \frac{(y-\alpha\sin\phi)^2 + z^2}{R^2}\left[\frac{1}{R} + \frac{6}{R^3}(D-x)(D-\alpha\cos\phi)\right], \\
(V_y, V_z) &= \frac{(y-\alpha\sin\phi, z)}{R^3}\left[x - \alpha\cos\phi - \frac{6}{R^2}(D-x)(D-\alpha\cos\phi)(2D-x-\alpha\cos\phi)\right], \\
U_x &= \frac{y-\alpha\sin\phi}{R^3}\left[x-\alpha\cos\phi + \frac{6}{R^2}(D-x)(D-\alpha\cos\phi)(2D-x-\alpha\cos\phi)\right], \\
(U_y, U_z) &= (1,0)\left[\frac{1}{R} + \frac{2}{R^3}(D-x)(D-\alpha\cos\phi)\right] + (y-\alpha\sin\phi, z)\frac{y-\alpha\sin\phi}{R^3} \\
&\quad \times\left[1 - \frac{6}{R^2}(D-x)(D-\alpha\cos\phi)\right]
\end{aligned} \tag{6}
$$

and

$$
R^2 = (2D - x - \alpha\cos\phi)^2 + (y-\alpha\sin\phi)^2 + z^2 . \tag{7}
$$

For each orientation of the stokeslet, the axisymmetric geometry allows the above flow fields to be determined by Hankel transforms. When the disk conditions (3) are applied to $\boldsymbol{v}$, the resulting pair of integral equations is

$$
\begin{aligned}
&\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_0^1 \left\{ f(\alpha,\phi)\begin{bmatrix} \dfrac{1}{r} + \dfrac{(x-\alpha\cos\phi)^2}{r^3} - V_x \\ \dfrac{(x-\alpha\cos\phi)(y-\alpha\sin\phi)}{r^3} - V_y \end{bmatrix}_{z=0} \right. \\
&\qquad \left. + g(\alpha,\phi)\begin{bmatrix} \dfrac{(y-\alpha\sin\phi)(x-\alpha\cos\phi)}{r^3} - U_x \\ \dfrac{1}{r} + \dfrac{(y-\alpha\sin\phi)^2}{r^3} - U_y \end{bmatrix}_{z=0} \right\} \alpha\,\mathrm{d}\alpha\,\mathrm{d}\phi = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&\qquad\qquad (0 \le \rho \le 1, -\pi < \theta \le \pi) .
\end{aligned} \tag{8}
$$

Transformation of the integral equation

Evidently it is necessary to consider Fourier components of (8). The density functions have expansions

$$
f(\alpha,\phi) = \sum_{m=0}^{\infty} \varepsilon_m f_m(\alpha)\cos m\phi , \quad g(\alpha,\phi) = 2\sum_{m=1}^{\infty} g_m(\alpha)\sin m\phi \tag{9}
$$

($\varepsilon_0 = 1$, $\varepsilon_m = 2$ if $m \ge 1$) while the kernel functions arising directly from the stokeslets, i.e. those exhibiting singularities, can be written

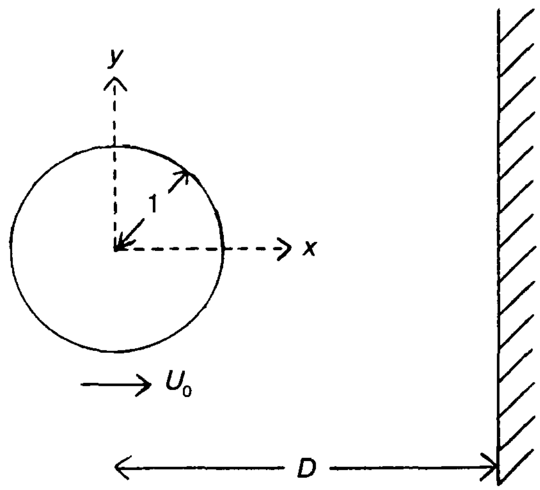

Fig. 1. The geometrical configuration.

$$\frac{1}{[\rho^2+\alpha^2-2\rho\alpha\cos(\theta-\phi)+z^2]^{1/2}}+\frac{\left(\rho\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\theta-\alpha\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\phi\right)^2}{[\rho^2+\alpha^2-2\rho\alpha\cos(\theta-\phi)+z^2]^{3/2}}$$

$$=\tfrac{1}{2}\left(3+z\frac{\partial}{\partial z}\right)[\rho^2+\alpha^2-2\rho\alpha\cos(\theta-\phi)+z^2]^{-1/2}$$

$$\pm\frac{\tfrac{1}{2}(\rho^2\cos 2\theta+\alpha^2\cos 2\phi)-\rho\alpha\cos(\theta+\phi)}{[\rho^2+\alpha^2-2\rho\alpha\cos(\theta-\phi)+z^2]^{3/2}},$$

$$\frac{(\rho\cos\theta-\alpha\cos\phi)(\rho\sin\theta-\alpha\sin\phi)}{[\rho^2+\alpha^2-2\rho\alpha\cos(\theta-\phi)+z^2]^{3/2}}=\frac{\tfrac{1}{2}(\rho^2\sin 2\theta+\alpha^2\sin 2\phi)-\rho\alpha\sin(\theta+\phi)}{[\rho^2+\alpha^2-2\rho\alpha\cos(\theta-\phi)+z^2]^{3/2}},$$

where

$$[\rho^2+\alpha^2-2\rho\alpha\cos(\theta-\alpha)+z^2]^{-1/2}=\sum_{m=0}^{\infty}\varepsilon_m\cos m(\theta-\phi)\int_0^{\infty}e^{-k|z|}J_m(k\rho)J_m(k\alpha)\,dk$$

and, by use of this expansion involving Bessel functions,

$$\frac{\rho^2\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}2\theta+\alpha^2\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}2\phi-2\rho\alpha\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(\theta+\phi)}{[\rho^2+\alpha^2-2\rho\alpha\cos(\theta-\phi)+z^2]^{3/2}}=-\frac{1}{z}\frac{\partial}{\partial z}\int_0^{\infty}e^{-k|z|}\,dk\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}$$

$$\Big\{[\rho^2J_m(k\rho)J_m(k\alpha)+\alpha^2J_{m+2}(k\rho)J_{m+2}(k\alpha)-2\rho\alpha J_{m+1}(k\rho)J_{m+1}(k\alpha)]\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}[(m+2)\theta-m\phi]$$

$$\pm[\rho^2J_m(k\rho)J_m(k\alpha)+\alpha^2J_{m-2}(k\rho)J_{m-2}(k\alpha)-2\rho\alpha J_{m-1}(k\rho)J_{m-1}(k\alpha)]\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}[(m-2)\theta-m\phi]\Big\}$$

But the k-integrals can be re-arranged, by integration by parts and use of the recurrence relations, to yield

$$\int_0^\infty e^{-k|z|}[\rho^2 J_m(k\rho)J_m(k\alpha) + \alpha^2 J_{m\pm 2}(k\rho)J_{m\pm 2}(k\alpha) - 2\rho\alpha J_{m\pm 1}(k\rho)J_{m\pm 1}(k\alpha)]\,dk$$
$$= \int_0^\infty e^{-k|z|} J_{m\pm 2}(k\rho)J_m(k\alpha)\left[|z|^2 + 3\frac{|z|}{k} + \frac{3}{k^2}\right]dk\,.$$

Then, since

$$-\frac{1}{z}\frac{\partial}{\partial z}\left[|z|^2 + 3\frac{|z|}{k} + \frac{3}{k^2}\right]e^{-k|z|} = (1 + k|z|)\,e^{-k|z|}\,,$$

the above formulae enable the integral equations (8), after substitution of (9), to be rearranged in the form

$$\begin{aligned}
&\sum_{n=0}^{\infty} \varepsilon_n \int_0^1 f_n(\alpha)\left\{\begin{bmatrix}3\\0\end{bmatrix}\cos n\theta \int_0^\infty J_n(k\alpha)J_n(k\rho)\,dk\right.\\
&\left.\quad + \tfrac{1}{2}\int_0^\infty J_n(k\alpha)\left[J_{n+2}(k\rho)\,{\cos \atop \sin}\,(n+2)\theta \pm J_{n-2}(k\rho)\,{\cos \atop \sin}\,(n-2)\theta\right]dk\right\}\alpha\,d\alpha\\
&\quad + 2\sum_{n=1}^{\infty}\int_0^1 g_n(\alpha)\left\{\begin{bmatrix}0\\3\end{bmatrix}\sin n\theta \int_0^\infty J_n(k\alpha)J_n(k\rho)\,dk\right.\\
&\left.\quad - \tfrac{1}{2}\int_0^\infty J_n(k\alpha)\left[J_{n+2}(k\rho)\,{\cos \atop \sin}\,(n+2)\theta \mp J_{n-2}(k\rho)\,{\cos \atop \sin}\,(n-2)\theta\right]dk\right\}\alpha\,d\alpha\\
&= \pi\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{\pi}\int_{-\pi}^{\pi}\int_0^1\left\{\begin{bmatrix}V_x\\V_y\end{bmatrix}_{z=0}\sum_{m=0}^{\infty}\varepsilon_m f_m(\alpha)\cos m\phi\right. \qquad (10)\\
&\quad \left. + \begin{bmatrix}U_x\\U_y\end{bmatrix}_{z=0}\sum_{m=0}^{\infty} 2g_m(\alpha)\sin m\phi\right\}\alpha\,d\alpha\,d\phi \qquad (0 \leqslant \rho \leqslant 1,\ -\pi < \theta \leqslant \pi)\,.
\end{aligned}$$

In previous work [6, 8, 9], integral equations of the second kind were obtained by using Abel transforms. Here the procedure needs to be applied to all Fourier modes by exploiting the identity [10]

$$\int_0^\infty J_n(k\alpha)J_n(k\rho)\,dk = \frac{2}{\pi}(\alpha\rho)^{-n}\int_0^{\min(\alpha,\rho)} \frac{s^{2n}\,ds}{(\alpha^2 - s^2)^{1/2}(\rho^2 - s^2)^{1/2}} \qquad (n \geqslant 0)$$

to define

$$[F_n(s), G_n(s)] = \frac{2}{\pi}\int_s^1 \left(\frac{s}{\alpha}\right)^n [f_n(\alpha), g_n(\alpha)]\,\frac{\alpha\,d\alpha}{(\alpha^2 - s^2)^{1/2}} \qquad (n \geqslant 0)\,, \qquad (11)$$

i.e.

$$[f_n(\alpha), g_n(\alpha)] = -\alpha^{n-1}\frac{d}{d\alpha}\int_\alpha^1 \frac{[F_n(s), G_n(s)]}{s^n}\,\frac{s\,ds}{(s^2 - \alpha^2)^{1/2}} \qquad (n \geqslant 0)\,. \qquad (12)$$

Then

$$\int_0^1 [f_n(\alpha), g_n(\alpha)]J_n(k\alpha)\alpha \, d\alpha = \int_0^1 [F_n(s), G_n(s)]h_{n-1}(ks)\, ds \qquad (n \geq 0)\,, \tag{13}$$

where

$$h_n(ks) = \frac{k}{s^n}\int_0^s \frac{\alpha^{n+1}J_n(k\alpha)}{(s^2-\alpha^2)^{1/2}}\, d\alpha = \left(\frac{\pi ks}{2}\right)^{1/2} J_{n+\frac{1}{2}}(ks) \qquad (n \geq -1)\,. \tag{14}$$

In particular

$$h_{-1}(ks) = \cos ks\,, \quad h_0(ks) = \sin ks\,.$$

The left hand sides of the integral equations (10) can now be expressed as

$$\int_0^1 F_0(s)\int_0^\infty h_{-1}(ks)\left\{\begin{bmatrix}3\\0\end{bmatrix}J_0(k\rho) + J_2(k\rho)\begin{matrix}\cos\\ \sin\end{matrix}\, 2\theta\right\} dk\, ds$$
$$+ 6\sum_{n=1}^\infty \int_0^1 \begin{bmatrix}F_n(s)\cos n\theta\\ G_n(s)\sin n\theta\end{bmatrix}\int_0^\infty h_{n-1}(ks)J_n(k\rho)\, dk\, ds$$
$$+ \sum_{n=1}^\infty \int_0^1\int_0^\infty h_{n-1}(ks)\Big\{[F_n(s) - G_n(s)]J_{n+2}(k\rho)\begin{matrix}\cos\\ \sin\end{matrix}\,(n+2)\theta$$
$$\pm [F_n(s) + G_n(s)]J_{n-2}(k\rho)\begin{matrix}\cos\\ \sin\end{matrix}\,(n-2)\theta\Big\} dk\, ds\,,$$

while on the right hand side, substitution of (12) yields

$$\int_0^1 \begin{bmatrix}v_x\\ v_y\end{bmatrix}_{z=0} f_0(\alpha)\alpha\, d\alpha = \begin{bmatrix}v_x\\ v_y\end{bmatrix}_{\substack{\alpha=0\\ z=0}} \int_0^1 F_0(s)\, ds + \int_0^1 F_0(s)\int_0^s \frac{\partial}{\partial\alpha}\begin{bmatrix}v_x\\ v_y\end{bmatrix}_{z=0} \frac{d\alpha}{(s^2-\alpha^2)^{1/2}}\, s\, ds\,,$$
$$\int_0^1 \begin{bmatrix}V_x\\ V_y\end{bmatrix} f_m(\alpha)\alpha\, d\alpha = \int_0^1 \frac{F_m(s)}{s^{m-1}}\int_0^s \frac{\partial}{\partial\alpha}\begin{bmatrix}\alpha^m V_x\\ \alpha^m V_y\end{bmatrix}_{z=0} \frac{d\alpha}{(s^2-\alpha^2)^{1/2}}\, ds \qquad (m \geq 1)\,, \tag{15}$$
$$\int_0^1 \begin{bmatrix}U_x\\ U_y\end{bmatrix} g_m(\alpha)\alpha\, d\alpha = \int_0^1 \frac{G_m(s)}{s^{m-1}}\int_0^s \frac{\partial}{\partial\alpha}\begin{bmatrix}\alpha^m U_x\\ \alpha^m U_y\end{bmatrix}_{z=0} \frac{d\alpha}{(s^2-\alpha^2)^{1/2}}\, ds \qquad (m \geq 1)\,.$$

Note that the regularity of the density functions implies, in (9), that f_n, g_n are $\mathcal{O}(\alpha^n)$ as $\alpha \to 0$, thus ensuring convergence of the integrals in (11), whence F_n, G_n are $\mathcal{O}(s^n)$ as $s \to 0$.

It is now necessary to consider Fourier components of (10) in order to exploit the identities

$$\frac{d}{dt}\int_0^t \frac{\rho^{n+1}J_n(k\rho)}{(t^2-\rho^2)^{1/2}}\, d\rho = t^n h_{n-1}(kt) \qquad (n \geq 0)\,, \tag{16}$$

derived from (14), and

$$\frac{2}{\pi}\int_0^\infty h_{n-1}(ks)h_{n-1}(kt)\,\mathrm{d}k = \delta(s-t) \qquad (n \geqslant 0)$$

$$\frac{2}{\pi}\int_0^\infty h_{n+1}(ks)h_{n-1}(kt)\,\mathrm{d}k = (2n+1)\,\frac{t^n}{s^{n+1}}\,H(s-t) - \delta(s-t) \qquad (n \geqslant 0) \tag{17}$$

obtained by substitution of (14) and suitable manipulation of the identity

$$\int_0^\infty J_\nu(ks)J_{\nu-1}(kt)\,\mathrm{d}k = \frac{t^{\nu-1}}{s^\nu}\,H(s-t)$$

given by [11, section 6.512]. Here $\delta(x)$ and $H(x)$ denote the Dirac delta and Heaviside unit functions respectively. Hence, by applying, for each $n \geqslant 0$, the operator

$$\frac{1}{\pi t^n}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho^{n+1}\,\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}} \quad (0 \leqslant t \leqslant 1)$$

to the nth Fourier components of the pair of integral equations of the first kind (10), the successive use of (13), (15), (16) and (17) yields an infinite set of integral equations of the second kind, namely

$$\begin{aligned}
&3F_0(t) + \int_t^1 \frac{F_2(s)+G_2(s)}{s}\,\mathrm{d}s - [F_2(t)+G_2(t)] = 2 + a_0(t)\,,\\
&2[F_1(t)+G_1(t)] = a_1(t) + b_1(t)\,,\\
&3[F_2(t)+G_2(t)] - F_0(t) + \frac{1}{t}\int_0^t F_0(s)\,\mathrm{d}s = a_2(t) + b_2(t)\,,\\
&3[F_n(t)+G_n(t)] - [F_{n-2}(t) - G_{n-2}(t)] + \frac{2n-3}{t^{n-1}}\int_0^t s^{n-2}[F_{n-2}(s) - G_{n-2}(s)]\,\mathrm{d}s\\
&\quad = a_n(t) + b_n(t) \qquad (n \geqslant 3)\,,\\
&3[F_n(t) - G_n(t)] - [F_{n+2}(t) + G_{n+2}(t)] + (2n+1)t^n\int_t^1 \frac{F_{n+2}(s)+G_{n+2}(s)}{s^{n+1}}\,\mathrm{d}s\\
&\quad = a_n(t) - b_n(t) \qquad (n \geqslant 1) \qquad (0 \leqslant t \leqslant 1)\,,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
\begin{bmatrix} a_n(t)(n \geqslant 0) \\ b_n(t)(n \geqslant 1) \end{bmatrix} &= \frac{1}{\pi^2}\int_{-\pi}^{\pi} \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta\,\mathrm{d}\theta\,\frac{1}{t^n}\,\frac{\partial}{\partial t}\int_0^t \frac{\rho^{n+1}\,\mathrm{d}\rho}{(t^2-\rho^2)^{1/2}}\Bigg[\Bigg[2\begin{bmatrix} V_x \\ V_y \end{bmatrix}_{\substack{\alpha=0\\ z=0}}\int_0^1 F_0(s)\,\mathrm{d}s\\
&+ \frac{1}{\pi}\int_{-\pi}^{\pi}\int_0^1\int_0^s \frac{\mathrm{d}\alpha}{(s^2-\alpha^2)^{1/2}}\,\frac{\partial}{\partial\alpha}\Bigg\{F_0(s)\begin{bmatrix} V_x \\ V_y \end{bmatrix}_{z=0} + 2\sum_{m=1}^{\infty}\left(\frac{\alpha}{s}\right)^m\Bigg\langle F_m(s)\begin{bmatrix} V_x \\ V_y \end{bmatrix}_{z=0}\cos m\phi\\
&+ G_m(s)\begin{bmatrix} U_x \\ U_y \end{bmatrix}_{z=0}\sin m\phi\Bigg\rangle\Bigg\}\Bigg]\Bigg]s\,\mathrm{d}s\,\mathrm{d}\phi \qquad (0 \leqslant t \leqslant 1)\,.
\end{aligned} \tag{19}$$

This is as far as the calculation can proceed without addressing the principal difficulty, namely, the evaluation of the double Fourier expansion of the reflected velocity fields at the disk.

In the absence of the plane ($D \to \infty$), the only non-zero inhomogenous term in (18) is 2 in the first equation and the system evidently has the solution

$$F_0(t) = \frac{2}{3}\,, \qquad F_n(t) = 0 = G_n(t) \qquad (n \geq 1) \qquad (0 \leq t \leq 1) \tag{20}$$

But, from (5), the force $F_D^* \hat{x}$ exerted by the disk on the fluid is given by

$$\begin{aligned} F_D^* &= \frac{8\mu U_0}{\pi} \int_{-\pi}^{\pi} \int_0^1 f(\alpha, \phi)\alpha \, \mathrm{d}\alpha \, \mathrm{d}\phi \\ &= 16 U_0 \mu \int_0^1 f_0(\alpha)\alpha \, \mathrm{d}\alpha \\ &= 16 U_0 \mu \int_0^1 F_0(s) \, \mathrm{d}s\,, \end{aligned}$$

after invoking (9) and (12). Thus, for unbounded fluid, the standard result $F_D^* = 32 U_0 \mu / 3$ (see e.g. [2] or [3]) is recovered and hence the dimensionless drag coefficient F_D should be defined by

$$F_D = \frac{3F_D^*}{32 U_0 \mu} = \tfrac{3}{2} \int_0^1 F_0(s) \, \mathrm{d}s\,. \tag{21}$$

4. Determination of the force coefficient to order $\mathbf{D^{-5}}$

The functions defined by (19) are evidently of the form

$$\begin{aligned} a_n(t) &= \int_0^1 \left\{ F_0(s) K_{0n}^x(s, t) + \sum_{m=1}^{\infty} [F_m(s) K_{mn}^x(s, t) + G_m(s) L_{mn}^x(s, t)] \right\} \mathrm{d}s\,, \quad (n \geq 0) \\ b_n(t) &= \int_0^1 \sum_{m=1}^{\infty} [F_m(s) K_{mn}^y(s, t) + G_m(s) L_{mn}^y(s, t)] \, \mathrm{d}s \qquad (n \geq 1)\,, \end{aligned} \tag{22}$$

and largely determine the kernel functions in the system (18). Since these cannot be determined from (6) is closed form, some combination of analytical and numerical approximation is required. A suitable choice is the expansion of the reflected velocities in powers of D^{-1} for this yields simple, separable kernels and automatically truncates the infinite system.

If only the terms $\frac{3}{2}D$, $\frac{1}{2}D$ in V_x, U_y respectively are retained, then the only contribution to (19) is a term $(4/\pi D)F_D$ to $a_0(t)$ and Brenner's result [12]

$$F_D \sim \left[1 - \frac{2}{\pi D}\right]^{-1}$$

is recovered.

The expansion of V_x, U_y is facilitated by observing, from (7), that

$$\frac{2}{R} + \frac{4}{R^3}(D - x)(D - \alpha \cos \phi) = \frac{3}{R} - \frac{r^2}{R^3}$$

and the algebra soon demonstrates that, for $n \geq 1$, $F_n(s)$ and $G_n(s)$ are of order $D^{-(n+1)}$.

Thus, for a solution up to order D^{-5}, only the kernels with $m = 0$ or 1 in (22) can enter the calculation. It is found, after lengthy manipulation, that

$$\pi a_0(t) = \left(\frac{6}{D} - \frac{3t^2}{2D^3} - \frac{41t^4}{64D^5}\right) \int_0^1 F_0(s)\,\mathrm{d}s - \left(\frac{3}{2D^3} - \frac{75t^2}{32D^5}\right) \int_0^1 s^2 F_0(s)\,\mathrm{d}s$$
$$- \frac{89}{64D^5} \int_0^1 s^4 F_0(s)\,\mathrm{d}s + \frac{3}{D^2} \int_0^1 s[2F_1(s) - G_1(s)]\,\mathrm{d}s\,,$$
$$\pi a_1(t) = \left(\frac{3t}{D^2} - \frac{5t^3}{4D^4}\right) \int_0^1 F_0(s)\,\mathrm{d}s - \frac{3t}{4D^4} \int_0^1 s^2 F_0(s)\,\mathrm{d}s$$
$$+ \frac{4t}{D^3} \int_0^1 s[2F_1(s) - G_1(s)]\,\mathrm{d}s\,,$$
$$\pi b_1(t) = -\frac{4t}{D^3} \int_0^1 s\left[F_1(s) - \frac{3}{8}\,G_1(s)\right]\mathrm{d}s\,,$$
$$\pi a_2(t) = \left(\frac{7t^2}{3D^3} - \frac{81t^4}{80D^5}\right) \int_0^1 F_0(s)\,\mathrm{d}s - \frac{23t^2}{16D^5} \int_0^1 s^2 F_0(s)\,\mathrm{d}s\,,$$
$$\pi[a_3(t), a_4(t)] = \left(\frac{3t^3}{2D^4}, \frac{51t^4}{56D^5}\right) \int_0^1 F_0(s)\,\mathrm{d}s\,,$$

with all others zero. Inspection of (18) then shows that the unknown functions must be polynomials, viz.

$$\left.\begin{aligned} &F_0(t) = A_0 + C_0 t^2 + E_0 t^4\,, \\ &\begin{bmatrix} F_1(t) \\ G_1(t) \end{bmatrix} = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} t + \begin{bmatrix} C_1 \\ D_1 \end{bmatrix} t^3\,, \quad \begin{bmatrix} F_3(t) \\ G_3(t) \end{bmatrix} = \begin{bmatrix} A_3 \\ B_3 \end{bmatrix} t^3 \\ &\begin{bmatrix} F_2(t) \\ F_1(t) \end{bmatrix} = \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} t^2 + \begin{bmatrix} C_2 \\ D_2 \end{bmatrix} t^4\,, \quad \begin{bmatrix} F_4(t) \\ G_4(t) \end{bmatrix} = \begin{bmatrix} A_4 \\ B_4 \end{bmatrix} t^4 \end{aligned}\right\}, \tag{23}$$

with all others zero, in the current approximation. Substitution of (23) into the left hand sides of equations (18) enables the coefficients to be determined in terms of the integrals appearing in the forcing functions. The first equation suffices to show, with F_D defined by (21), that

$$\begin{aligned} 2F_D &= 3A_0 + C_0 + \frac{3}{5}\,E_0 \\ &= 2 + \frac{F_D}{\pi}\left[\frac{4}{D} - \frac{1}{3D^3} - \frac{41}{320D^3}\right] - \frac{1}{\pi}\left[\frac{3}{2D^3} - \frac{25}{32D^3}\right] \int_0^1 s^2 F_0(s)\,\mathrm{d}s \\ &\quad + \frac{3}{\pi D^2} \int_0^1 s[2F_1(s) - G_1(s)]\,\mathrm{d}s - \frac{89}{64\pi D^5} \int_0^1 s^4 F_0(s)\,\mathrm{d}s\,, \end{aligned} \tag{24}$$

and the third equation confirms that $C_0 = O(D^{-3})$, $E_0 = O(D^{-5})$. Then substitution of (23) gives

$$\int_0^1 s^2 F_0(s)\,\mathrm{d}s = \frac{1}{3}A_0 + \frac{1}{5}C_0 + \frac{1}{7}E_0 = \frac{2}{9}F_D + O(D^{-3}),$$

$$\int_0^1 s^4 F_0(s)\,\mathrm{d}s = \frac{2}{15}F_D + O(D^{-3}),$$

$$\int_0^1 s[2F_1(s) - G_1(s)]\,\mathrm{d}s = \frac{1}{3}(2A_1 - B_1) + \frac{1}{5}(2C_1 - D_1).$$

But the second and last ($n = 1$) equations in (18) yield

$$A_1 + B_1 = \frac{F_D}{\pi D^2} + O(D^{-4}),\ A_1 - B_1 = \frac{2F_D}{3\pi D^2} + O(D^{-4}),$$
$$C_1, D_1 = O(D^{-4}),$$

and hence, without the need to consider further equations in (18) or determine other coefficients, the above results suffice, on substitution in (24), to show that

$$F_D^{-1} = 1 - \frac{2}{\pi D} + \frac{1}{3\pi D^3} - \frac{3}{4\pi^2 D^4} + \frac{7}{144\pi D^5} + O\left(\frac{1}{D^6}\right). \tag{25}$$

Note that the second Fourier mode contribution is $O(D^{-3})$ while that of the first mode is $O(D^{-4})$.

The velocity fields can, in principle, be determined by substitution of (23) in (12) and then (5) but the calculation is lengthy, depending on the accuracy required, even on the axis $\rho = 0$ of the disk.

The uniform validity of the expansions in inverse powers of R, employed to obtain the asymptotic estimate (25), requires that D exceed unity. The alternating signs indicate that the mathematical error is less than the first term neglected. As D is decreased toward unity, only the quasi-static approximation is threatened because, in contrast to the broadside approach, no squeezing of the fluid between disk and plane can occur.

In the experimental work of Trahan and Hussey [1], disks of radii 0.5 or 1 cm and various thicknesses down to 0.5 mm are used and thus lubrication effects are never eliminated. It is not obvious how best to extrapolate the experimental values of the force coefficient to disks of zero thickness in order to make a valid comparison with the analytical approximation (25). Currently, the experiments yield a shortfall that increases from 0.020 at $D^{-1} = 0.4$ to 0.035 at $D^{-1} = 0.8$.

References

1. J.F. Trahan and R.G. Hussey, private communication.
2. K.B. Ranger, The circular disk straddling the interface of a two phase flow. *Intl. J. Multiphase Flow* 4 (1978) 263–277.
3. A.M.J. Davis, Slow viscous flow due to motion of an annular disk; pressure-driven extrusion through an annular hole in a wall. *J. Fluid Mech.* 231 (1991) 51–71.
4. S.H. Smith, Stokes flows past slits and holes. *Int. J. Multiphase Flow* 13 (1987) 219–231.
5. A.M.J. Davis, Shear flow disturbance due to a hole in the plane. *Phys. Fluids* A3 (1991) 478–480.
6. A.M.J. Davis, Stokes drag on a disk sedimenting toward a plane or with other disks; additional effects of a side wall or free surface. *Phys Fluids* A2 (1990) 301–312.

7. J.R. Blake, A note on the image system for a stokeslet in a no-slip boundary. *Proc. Cambridge Philos. Soc.* 70 (1971) 303–310.
8. A.M.J. Davis, A translating disk in a Sampson flow; pressure-driven flow through concentric holes in parallel walls. *Q.J. Mech. Appl. Math.* 44 (1991) 471–486.
9. A.M.J. Davis, Creeping flow through an annular stenosis in a pipe, *Q. Appl. Math.* 49 (1991) 507–520.
10. J.C. Cooke, Triple integral equations. *Q.J. Mech. Appl. Math.* 16 (1963) 193–203.
11. I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products* (enlarged edition, ed. A. Jeffrey). Academic (1980).
12. H. Brenner, Effect of finite boundaries on the Stokes resistance of an arbitrary particle. *J. Fluid Mech.* 12 (1962) 35–48.